Math 821 Lecture 6

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Stirling's approximation

You've probably seen Stirling's approximation at some point in your education, but you may not have seen (or remember) the proof. For the first part of today, we'll prove Stirling's formula[1] so we can do some asymptotics in the second part.

[1] well, we'll prove it up to the constant factor $\sqrt{2\pi}$. That part isn't so interesting to us right now anyways.

Proposition (Stirling's formula)**.** *As $n \to \infty$,*

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Proof (mostly). First notice that the factorial/exponential part just comes from an integral estimate. How? Notice that $\log n! = \log 1 + \log 2 + \cdots + \log n$ and

$$\int_0^{n-1} \log t\, dt \leq \log 1 + \log 2 + \cdots + \log(n-1) \leq \int_1^n \log t\, dt$$

The antiderivative of log is $x \log x - x$, so this tells us that

$$\begin{aligned}(n-1)\log(n-1) - (n-1) - \lim_{\epsilon \to 0}(\epsilon \log \epsilon - \epsilon) \\ \leq \log(n-1)! \\ \leq n \log n - n - (\log 1 - 1),\end{aligned}$$

or in other words, that

$$(n-1)\log(n-1) - (n-1) \leq \log(n-1)! \leq n \log n - n + 1.$$

If we take the exp of everything in the above inequalities, we get

$$\frac{(n-1)^{n-1}}{e^{n-1}} \leq (n-1)! \leq e \cdot \frac{n^n}{e^n}$$

so we have the exponential growth rate correct.

Next we need to look at the lower order factor(s). Let

$$d_n = \log n! - ((n + 1/2)\log n - n)\,;$$

the thing in the brackets is just the log of $\sqrt{n}(n/e)^n$. We would like to show that d_n approaches a constant c, as that would show that $n!$ and $\sqrt{n}(n/e)^n$ asymptotically differ only by that constant factor.

Let's simplify $d_n - d_{n+1}$

$$\begin{aligned}&-\log(n+1) - (n + 1/2)\log n + n + (n + 1 + 1/2)\log(n+1) - (n+1) \\ &= (n + 1/2)\log\left(\frac{n+1}{n}\right) - 1,\end{aligned}$$

then re-complicate it with

$$\left(\frac{n+1}{n}\right) = \left(\frac{1+\frac{1}{2n+1}}{1-\frac{1}{2n+1}}\right)$$

so we can use the following (less common) Taylor expansion of log:

$$\frac{1}{2}\log\left(\frac{1+t}{1-t}\right) = t + \frac{1}{3}t^3 + \frac{1}{5}t^5 + \cdots .$$

This gives

$$\begin{aligned}
d_n - d_{n+1} &= \left(\frac{2n+1}{2}\right)\log\left(\frac{1+\frac{1}{2n+1}}{1-\frac{1}{2n+1}}\right) - 1 \\
&= 1 + \frac{1}{3(2n+1)^2} + \frac{1}{5(2n+1)^4} + \cdots \\
&\leq \frac{1}{3}\left(\frac{1}{(2n+1)^2}) + \frac{1}{(2n+1)^4} + \cdots\right) \\
&= \frac{1}{3}\frac{\left(\frac{1}{2n+1}\right)^2}{1-\left(\frac{1}{2n+1}\right)^2} \\
&= \frac{1}{3}\frac{1}{(2n+1)^2-1} \\
&= \frac{1}{3}\frac{1}{4n^2+4n} \\
&= \frac{1}{12}\frac{1}{n(n+1)}.
\end{aligned}$$

We conclude that $\{d_n\}$ is decreasing, but $\{d_n - \frac{1}{12n}\}$ is increasing. It follows that both sequences converge to the same (finite) limit c, and hence

$$n! \approx e^c\sqrt{n}\left(\frac{n}{e}\right)^n,$$

which is what we set out to prove[2]. □

[2] There's a few ways to properly finish this and get the constant $e^c = \sqrt{2\pi}$. One way is through Wallace's product formula

$$\frac{\pi}{2} = \prod_{n=1}^{\infty}\frac{(2n)(2n)}{(2n-1)(2n+1)}.$$

We've spent long enough on this digression—you probably didn't sign up for combinatorics to learn analysis! So let's move on to some rooted trees.

Pólya's analysis of rooted trees

We've studied a few different combinatorial constructions so far. Quite a few of them have similar asymptotics[3]; in this section, we'll try to find them.

[3] so many that the asymptotic forms are sometimes called "universal laws"

Consider the combinatorial class of rooted non-planar unlabelled trees $\mathcal{T} = \mathcal{Z} \times \textsc{MSet}(\mathcal{T})$. This has generating function

$$T(x) = x\exp\left(\sum_{n=1}^{\infty}\frac{T(x^n)}{n}\right)$$

which determines $T(x)$ recursively. We'll apply some analysis to this generating function to get some asymptotic information about the coefficients t_n. But this means we'll actually have to care about $T(x)$ as a function—not just as a formal power series—and worry about stuff like the radius of convergence.

Combinatorially, we know that there's at least one rooted tree of every size[4]. So $t_n \geq 1$ for all $n \geq 1$, and the radius of convergence ρ of $T(x)$ is at most 1. Also,

$$T(x) \geq x\frac{T(x)^2}{2} \tag{1}$$

since all coefficients t_n are positive[5], and $1 \geq xT(x)/2$. As x approaches ρ from the left, we get

$$1 \geq \rho\frac{\lim_{x\to\rho^-} T(x)}{2},$$

so $T(\rho) < \infty$.

[4] except size zero, but the first term doesn't affect the radius of convergence

[5] the right hand side comes from the second term of the Taylor expansion of exp, not from the interior sum which has the power ^{n} on the inside.

But wait—what if $\rho = 0$? This would be really bad if we're trying to apply analytic techniques to $T(x)$ around $x = 0$. In the next step, we show that this doesn't happen. First, some definitions:

Definition. *Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0} b_n x^n$. We write*

$$A(x) \trianglelefteq B(x)$$

if $a_n \leq b_n$ for all $n \geq 0$.

Definition. *If there exists a $R > 0$ such that $A(x) \trianglelefteq \sum_{j=0}^{\infty} R^j x^j$, we say that $A(x)$ is* bounded*.*

Definition. *Let Φ be an admissible combinatorial construction and Θ the associated operator on power series. We say Θ is* bounded *if there exists a constant $R > 0$ such that for all $A(X) \in x\mathbb{R}^{\geq 0}[[x]]$,*

$$\Theta(A(x) \trianglelefteq \sum_{j=0}^{\infty} R^j (x + A(x))^j$$

Observe that the operator on power series associated with MSET is bounded in this sense. Why? The generating function for MSET is

$$\exp\left(\sum_{n=1}^{\infty} \frac{A(x^n)}{n}\right) \trianglelefteq \frac{1}{1 - A(x)} = \sum_{j=0}^{\infty} A(x)^j \trianglelefteq \sum_{j=0}^{\infty} (x + A(x))^j$$

where the first $\trianglelefteq$ inequality comes from the combinatorial fact that $|(\textsc{MSet}(\mathcal{A}))_n| \leq |(\textsc{Seq}(\mathcal{A}))_n|$. This satisfies the above definition with $R = 1$.

Lemma. *Let Θ be a bounded power series operator and suppose $[x^n]\Theta(A(x))$ depends only on $[x^j]A(x)$ for $j < n$. Let $T(x)$ be the unique series defined by $T(x) = \Theta(T(x))$. Then $T(x)$ has positive radius of convergence.*

Proof. We know that for all $A(x)$,

$$\Theta(A(x)) \trianglelefteq \sum_{j=0}^{\infty} R^j (x + A(x))^j$$

so

$$[x^n]\Theta(A(x)) \le [x^n] \sum_{j=0}^{\infty} R^j (x + A(x))^j.$$

By assumption, the left hand side does not depend on $[x^n]A(x)$—it only depends on earlier terms of $A(x)$—so the inequality remains true even if we set $[x^n]A(x) = 0$. So

$$[x^n]\Theta(A(x)) \le [x^n] \sum_{j=0}^{\infty} R^j (x + A(x))^j - R[x^n]A(x)$$

$$\Theta(A(x)) \trianglelefteq \sum_{j=0}^{\infty} R^j (x + A(x))^j - RA(x).$$

By this, the solution to $T(x) = \Theta(T(x))$ is bounded by the solution to

$$(1 + R)S(x) = \sum_{j=0}^{\infty} R^j (x + S(x))^j \tag{2}$$

Since the right hand side is equal to $\frac{1}{1-Rx-RS(x)}$, we can rewrite equation (2) as

$$(1 + R)S(x)(1 - Rx - RS(x)) = x + S(x)$$

which is quadratic in $S(x)$. Checking the discriminant $((1 + R)(1 - Rx) - 1)^2 - 4x(1 + R)R$ at $x = 0$ and $R^2 > 0$, we find that the solution $S(x)$ exists as an analytic function near 0. So $S(x)$ has positive radius of convergence, and hence so does $T(x)$. □

Corollary. *If $\mathcal{T}$ is the class of rooted trees, then $T(x)$ has radius of convergence ρ with $0 < \rho < 1$ and $T(\rho) < \infty$.*

All the above was just fussing around—mere prerequisites for the real trick which is to come. Here is the real meat of this lecture: define

$$E(x, y) = xe^y \exp\left(\sum_{n=2}^{\infty} \frac{T(x^n)}{n}\right)$$

by taking the leading term[6] out of $T(x)$ such that $T(x) = E(x, T(x))$. Since ρ is smaller than 1, $\rho > \rho^2 > \rho^3 > \cdots$ and thus we can choose $\epsilon > 0$ such that

$$\exp\left(\sum_{n=2}^{\infty} \frac{T((\rho + \epsilon)^n)}{n}\right) < \infty$$

so $E(\rho + \epsilon, T(\rho) + \epsilon)$ exists and $E(x, y)$ is analytic in a neighbourhood of $(\rho, T(\rho))$. This is the key: it lets us analyse these formulas more easily.

Here's a theorem we'll prove next time to do this analysis.

[6] referring to the expansion

$$T(x) = x \exp\left(\sum_{n=1}^{\infty} \frac{T(x^n)}{n}\right)$$

from equation (1)

Theorem. *Suppose* $T(x) \in x\mathbb{R}^{\geq 0}[[x]]$ *and* $E(x,y) \in \mathbb{R}^{\geq 0}[[x,y]]$ *with*

- $E(0,0) = 0$
- *E has a term of degree* ≥ 2 *in y*
- $\frac{\partial}{\partial x} E(x,y) \neq 0$*, and*
- $T(x) = E(x, T(x))$.

Let ρ *be the radius of convergence of* $T(x)$*. Suppose that* $0 < \rho < \infty$*, that* $T(\rho) < \infty$ *and that there is some* $\epsilon > 0$ *such that* $E(\rho + \epsilon, T(\rho + \epsilon))$ *exists. Then there are functions* $A(x), B(x)$ *analytic at* 0 *such that*

$$T(x) = A(\rho - x) + B(\rho - x)\sqrt{\rho - x}$$

for $|x| < \rho$*, x near* ρ.

This theorem lets us describe T near the singularity (which is essentially a square root singularity). As $x \to \rho$, $T(x) \approx A(0) + B(0)\sqrt{\rho - x}$ and we know

$$\begin{aligned}
[x^n]\sqrt{\rho - x} &= \binom{1/2}{n}\rho^{1/2 - n}(-1)^n \\
&\approx \rho^{\frac{1}{2} - n} n^{-\frac{1}{2} - 1} \cdot c \\
&= \frac{1}{\rho^n} n^{-3/2} c'
\end{aligned}$$

So t_n has an exponential part and a “lower order part” which is always to the power of $-3/2$.

What remains to be done? We need a “transfer theorem" to show when $A(x) \approx B(x)$, then $a_n \approx b_n$. We need to prove the above theorem, of course, and then we need to build examples of families with our operators for which this whole story holds. Next week, we’ll finish this.

References

[1] J. P. Bell, S. N. Burris, and K. A. Yeats. Counting Rooted Trees: The Universal Law $t(n) \approx C\rho^{-n}n^{-3/2}$. *arXiv preprint*, 2005.